Gain-Sensitivity Augmentation for Near-Optimal Control of Linear Parameter-Dependent Plants

Peter L. Graf* and Rahmat Shoureshi†
Purdue University, West Lafayette, Indiana 47907

This paper develops a parameter-adaptive version of the steady-state linear quadratic Gaussian controller for plants with structured parameter dependencies. Both scalar and vector parameter perturbations are treated. The design method, referred to as gain-sensitivity augmentation, is based on approximating the optimal parameter-dependent regulator and filter gain matrices with truncated Taylor series expansions. The coefficient matrices of the Taylor series expansions are referred to as gain-sensitivity matrices and are precomputed off-line. Parameter information, assumed to be available on-line by direct measurement or estimation, is used to adjust the gain matrices to near-optimal values.

Introduction

IN practice, many systems may be adequately represented for control design by multi-input/multi-output (MIMO) time-invariant linear models. Often, actual system parameters may vary significantly from modeled values, degrading control system performance, or even initiating instabilities. A major portion of the current research effort in control system design is focused on parameter robustness, that is, desensitizing control system performance to variations in plant parameters.

Parameter perturbations in linear state-space models are generally categorized as unstructured or structured perturbations of the coefficient matrices. Structured perturbations refer to element level variations, whereas unstructured perturbations refer to matrix level variations. Parameter perturbations may be further classified as deterministic or stochastic. This paper addresses only the class of structured deterministic parameter perturbations.

Examples of some methods for robust control design for structured parameter perturbations are the guaranteed cost control of Vinkler and Wood,[1] trajectory sensitivity control design of Yedavalli and Skelton,[2] maximum entropy control design of Hyland and Bernstein,[3] and gain scheduling.[4]

Trajectory-sensitivity control design represents an extension to the linear quadratic Gaussian (LQG) controller design method. The method assumes explicit knowledge of the structure of parameter dependencies in the system matrices; however, the parameter dependencies are assumed uncertain. The trajectory sensitivity system consists of the nominal plant model augmented by a trajectory-sensitivity subsystem. The controller design utilizes feedback from both nominal plant states and sensitivity states. No optimal constant-gain feedback law exists for this problem; however, several suboptimal designs have been proposed. The main difficulty with this approach is the high order of the resulting controllers.

Guaranteed cost control is a method for synthesizing a closed-loop system in which the controlled plant has large parameter uncertainties. The method uses a multistep algorithm to choose constant feedback gains that insure stable closed-loop behavior for a potentially large range of parameter values. A restriction of the method is the assumption that the parameters enter the A matrix linearly. Full-state feedback is assumed. The method tends to produce feedback gains that result in overdamping of the dominant closed-loop poles.

Maximum entropy control design addresses structured parameter perturbations from a stochastic point of view. The formulation includes multiplicative noise in control, state, and measurement matrices to account for the maximum uncertainty in the plant. The justification for this uncertainty model is given by appealing to Jaynes' maximum entropy principle. Using the Fisk-Stratonovich interpretation of stochastic integration, necessary conditions to minimize a linear quadratic performance metric more general than the standard LQG are derived. The necessary conditions appear as four nonlinear matrix equations coupled by stochastic effects. The resulting controllers exhibit improved robustness to parameter variations in the plant matrices.

Gain scheduling is the oldest and most popular technique for accommodating plant parameter variations in control design. In gain scheduling, auxiliary measurements of critical parameters are used to select new regulator gains. Usually, implementation consists of a schedule of gains indexed by the measured parameters. Its simplicity and effectiveness make it the "predominant adaptive scheme used in advanced avionic systems to achieve high performance."[4] Gain scheduling is an effective and proven method for compensating for plants with one or two significant parameter variations; however, the method quickly becomes cumbersome with additional parameter variations.

This paper considers the gain-sensitivity augmentation approach first proposed for the steady-state linear-quadratic state feedback (LQSF) regulator by Barnett and Storey[5] for the single parameter case and extended to the multiple parameter case by Vetter et al.[6] In both cases, only the first-order gain-sensitivity augmentation was developed. We extend this technique to the steady-state LQG regulator. Only the continuous time case is considered. Both scalar and vector plant parameter dependencies are treated. The gain-sensitivity equations are generalized to permit augmentation to an arbitrary order to provide robust control in the presence of very large parameter perturbations. It is assumed that the structure of parameter perturbations in the plant is known, as well as the perturbation bounds for each parameter. Parameters are assumed to be available from either direct measurement or estimation scheme. The recursive gain-sensitivity equations are derived and presented in an algorithmic form, which simplifies computer implementation.

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* Ray W. Herrick Laboratories, School of Mechanical Engineering; currently with Space and Communication Group, Hughes Aircraft Co., Los Angeles, CA.

† Professor, Ray W. Herrick Laboratories, School of Mechanical Engineering.

A vibration control example is used to illustrate the control design. In this example, a harmonic disturbance of unknown amplitude and parameterized by a measurable frequency is accomodated using gain-sensitivity augmentation in a disturbance-utilization mode.[7]

Notation and Conventions

Matrices are represented by capital letters. Vectors are represented by lowercase letters. The symbol α denotes a scalar parameter whereas boldface $\boldsymbol{\alpha}$ denotes a parameter vector. The following notation will often be used to denote the derivative of a matrix with respect to a scalar parameter.

$$\frac{d^q}{d\alpha^q} A(\alpha) \triangleq A^{(q)}(\alpha) \tag{1}$$

The vector derivative of a matrix is defined as

$$D_{\alpha(i)}\left[A(\alpha)\right] \triangleq \underbrace{\frac{d}{d\alpha}\cdot\frac{d}{d\alpha}\cdots\frac{d}{d\alpha}}_{i \text{ times}}\left[A(\alpha)\right] \tag{2}$$

An n-term Taylor series expansion of a scalar-valued function is denoted by

$$^{n}G(\alpha) \triangleq G(\alpha_0) + \sum_{i=1}^{n} G^{(i)}(\alpha)\,\frac{(\alpha-\alpha_o)^i}{i!} \tag{3}$$

The Kronecker product or direct matrix product of matrices A and B is formed by multiplying every element of A by the entire matrix B.[8]

$$\underset{p\times q}{A} \otimes \underset{s\times t}{B} = \langle a_{ij}B\rangle \quad \begin{matrix} i\rightarrow = 1, 2\cdots p \\ j\downarrow = 1, 2\cdots q \end{matrix} \tag{4}$$

The logic expression

$$\left\{q \overset{?}{=} r\right\} \tag{5}$$

has a value of 1 if it is true, 0 if it is false.

Problem Formulation

This paper considers linear multi-input/multi-output (MIMO) parameter-dependent time-invariant systems of finite order disturbed by Gaussian distributed white noise. The measurements are presumed to be corrupted by Gaussian distributed white noise uncorrelated with the plant noise. The presumed form of the plant is

$$\dot{x}(t,\alpha) = A(\alpha)x(t,\alpha) + B(\alpha)u(t,\alpha) + Dw(t) \tag{6a}$$

$$y(t,\alpha) = C(\alpha)x(t,\alpha) \tag{6b}$$

$$z(t,\alpha) = M(\alpha)x(t,\alpha) + v(t) \tag{6c}$$

The performance metric for the control design of the system is chosen as

$$\mathcal{V} = \lim_{t\rightarrow\infty} \frac{1}{t} E\left\{\int_0^t y^T(t,\alpha)Qy(t,\alpha) + u^T(t,\alpha)Ru(t,\alpha)dt\right\} \tag{7}$$

where x, y, $z \in \mathbf{R}^{n_x\times 1}$, $\mathbf{R}^{n_y\times 1}$, $\mathbf{R}^{n_z\times 1}$, respectively, and $\boldsymbol{\alpha} \in \mathbf{R}^{p\times 1}$. Q and R are positive-definite weighting matrices, w and v are zero-mean Gaussian distributed white noise with covariances W and V, respectively.

The following assumptions are made:

1) Parameter dependence is confined to the A, B, C, M plant matrices.

2) The admissible parameter perturbations are elements of an open set with prescribed bounds, i.e., $\boldsymbol{\alpha} \in \Omega$.

3) The system maintains stabilizability of the pairs $[A(\alpha),B(\alpha)]$,$[A(\alpha),D]$ and detectability of the pairs $[A(\alpha),C(\alpha)]$,$[A(\alpha),M(\alpha)]$ for all admissible parameter perturbations.

4) The functional dependence of the plant matrices on the parameters is known and is not explicit in time.

5) The plant matrices A, B, C, M are continuous and infinitely differentiable with respect to each parameter over the admissible set of parameter perturbations.

The problem is to synthesize a controller that stabilizes the system and maintains optimal or near-optimal performance relative to the criterion of Eq. (7) in the presence of unpredictable, but measurable or estimatable parameter perturbations.

Optimal Parameter-Dependent LQG Controller

Theorem 1 shows that the optimal steady-state feedback compensator for the parameter-dependent plant incorporates a parameter-dependent form of the well-known Kalman filter.

Theorem 1: The optimal steady-state feedback law for the system given in Eqs. (6a–6c) that minimizes the performance metric in Eq. (7) is

$$u(t,\alpha) = G(\alpha)x_c(t,\alpha) \tag{8a}$$

where the controller states x_c given by the parameter-dependent Kalman filter

$$\dot{x}_c(t,\alpha) = A(\alpha)x_c(t,\alpha) + B(\alpha)u(t,\alpha) + F(\alpha)M(\alpha)[x(t,\alpha) - x_c(t,\alpha)] \tag{8b}$$

The functional dependence of the gain matrices F(α) and G(α) is summarized as follows:

$$G(\alpha) = -R^{-1}B^T(\alpha)K(\alpha) \tag{8c}$$

$$F(\alpha) = P(\alpha)M^T(\alpha)V^{-1} \tag{8d}$$

$$0 = K(\alpha)A(\alpha) + A^T(\alpha)K(\alpha) - K(\alpha)B(\alpha)R^{-1}B^T(\alpha) + C^T(\alpha)QC(\alpha) \tag{8e}$$

$$0 = P(\alpha)A^T(\alpha) + A(\alpha)P(\alpha) - P(\alpha)M^T(\alpha)V^{-1}M(\alpha)P(\alpha) + DWD^T \tag{8f}$$

The optimal solution to the parameter-dependent LQG problem for the stated assumptions is essentially equivalent to that of the standard LQG problem. Derivations for the LQG controller are available in numerous texts (e.g., Ref. 9). Because the result of the LQG problem is well known, only an outline of a proof is given in the Appendix.

Proposed Control Design

Equations (8a–8f) reveal that the optimal control policy for the parameter-dependent system is an implicit function of $\boldsymbol{\alpha}$. The dependence is tied up in both the regulator gain matrix and the controller states. Assume that a convergent Taylor series expansion exists for the control $u(t,\boldsymbol{\alpha})$ in Eq. (8a) w.r.t. $\boldsymbol{\alpha}$ about some nominal vector $\boldsymbol{\alpha}_o$. For simplicity, we will assume $\alpha \in \mathbf{R}^{1\times 1}$. Then,

$$u(t,\alpha) = u(t,\alpha_o) + u'(t,\alpha_o)\Delta\alpha + u''(t,\alpha_o)(\Delta\alpha^2/2!) + \cdots \tag{9}$$

Substitution for u results in

$$u(t,\alpha) = \{(Gx_c + G'x_c + Gx_c')\Delta\alpha + (G''x_c + 2G'x_c' + Gx_c'')(\Delta\alpha^2/2!) + \cdots\}_{\alpha=\alpha_o} \tag{10}$$

After some rearrangement, Eq. (10) can be written in terms of an expansion of G and x_c about α_o. However, the x_c portion of this expansion can be substituted by the exact $x_c(t,\alpha)$; therefore, the following can be obtained:

$$u(t,\alpha) = (G + G'\Delta\alpha + G''(\Delta\alpha^2/2!) + \cdots)|_{\alpha=\alpha_o} x_c(t,\alpha) \quad (11)$$

Equation (11) indicates that the control $u(t,\alpha)$ may be expressed as the product of the Taylor series expansion for the gain matrix expanded about α_o and the states of the optimal parameter-dependent LQG controller. To recover the optimal control policy, the parameter-dependent controller states $x_c(t,\alpha)$ and the regulator gain $G(\alpha)$ must be known exactly. The controller states are a function of the plant matrices $A(\alpha)$, $B(\alpha)$, $M(\alpha)$, and the filter gain $F(\alpha)$.

We now invoke the assumption that the parameter information is available from either direct measurement or estimation. Because we know the functional dependence of the plant matrices on the parameters, they may be recovered exactly. The difficulty arises in computation of $F(\alpha)$ and $G(\alpha)$. Equations (8c) and (8d) show that the gain matrices $F(\alpha)$ and $G(\alpha)$ are functions of Riccati matrices $P(\alpha)$ and $K(\alpha)$, respectively. Because of the nonlinear structure of the Riccati equations (8e) and (8f), closed-form solutions for $P(\alpha)$ and $K(\alpha)$ do not exist.

The preceding discussion implies the need to approximate the gain matrices. The proposed approach is to approximate the gain matrix functions $F(\alpha)$ and $G(\alpha)$ by an n-term truncated Taylor series expansion about a nominal vector α_o. The form of the proposed GSA-LQG controller is

$$\hat{u}(t,\alpha) = {}^nG(\alpha)\hat{x}_c(t,\alpha) \quad (12a)$$

$$\dot{\hat{x}}_c(t,\alpha) = A(\alpha)\hat{x}_c(t,\alpha) + B(\alpha)\hat{u}(t,\alpha) + {}^nF(\alpha)M(\alpha)[x(t,\alpha) - \hat{x}_c(t,\alpha)] \quad (12b)$$

${}^nF(\alpha)$ and ${}^nG(\alpha)$ are n-term Taylor series expansions of the gain matrices.

Gain-Sensitivity Augmentation Technique

In this section, the GSA technique is extended to the steady-state stochastic regulator (LQG) design. Both scalar and vector plant parameter dependencies are treated. The GSA technique is generalized to permit gain matrix augmentation to an arbitrary order.

Scalar Case

Consider the simple case in which the plant is dependent on only the single parameter α in the A matrix. Assuming the existence of a convergent Taylor series, the expansion for the regulator gain matrix becomes

$$\infty G(\alpha) = \sum_{i=0}^{\infty} G^{(i)}(\alpha)\bigg|_{\alpha=\alpha_o} \frac{(\alpha-\alpha_o)^i}{i!} \quad (13a)$$

$$= -R^{-1}B^T\sum_{i=0}^{\infty} K^{(i)}\frac{(\alpha)}{i!}\bigg|_{\alpha=\alpha_o} \frac{(\alpha-\alpha_o)^i}{i!} \quad (13b)$$

The gain matrix expansion is a function of the Riccati sensitivity matrices. Differentiating the regulator Riccati matrix equation (8e) one time yields an equation that admits a standard Lyapunov form for $K^{(1)}(\alpha)$.

$$K^{(1)}(\alpha)|_{\alpha=\alpha_o}L(\alpha_o) + L^T(\alpha_o)K^{(1)}(\alpha)|_{\alpha=\alpha_o} = -[K(\alpha)A^{(1)}(\alpha) + A^{T(1)}(\alpha)K(\alpha)]|_{\alpha=\alpha_o} \quad (14)$$

where

$$L(\alpha) \triangleq A(\alpha) + B(\alpha)G(\alpha) \quad (15)$$

Repeated differentiation of the Riccati equation will yield Lyapunov equations with the highest-order sensitivity matrices appearing as the unknown Lyapunov matrices. A compact expression for the Riccati sensitivity equations may be derived by application of the following recursive product differentiation formula.

Recursive Product Rule 1 (Scalar Differentiation of a Product of Matrices)

Consider the derivative of the matrix product $A(\alpha)B(\alpha)$.

$$\frac{d^k}{d\alpha^k}\Big[A(\alpha)B(\alpha)\Big] \triangleq D_\alpha^{(k)}\Big[A(\alpha)B(\alpha)\Big] \quad (16)$$

where $A(\alpha)$ and $B(\alpha)$ are conformable for multiplication, α is a scalar, and $A(\alpha)$, $B(\alpha)$ are continuous and infinitely differentiable functions of α. Then, we may write a recursive algorithm for repeated differentiation as

$$D_\alpha^{(n)}[A(\alpha)B(\alpha)] = \sum_{q=1}^{n}\binom{n-1}{q-1}A^{(q)}B^{(n-q)} + A^{(q-1)}B^{(n-q+1)} \quad (17)$$

The recursive Riccati sensitivity equation for the general scalar case where A, B, C, and M are functions of the parameter α, as derived using Eq. (8e) in conjunction with Eq. (17), is given as

$$\begin{aligned} K^{(q)}L + L^TK^{(q)} = \sum_{q=1}^{n-1}\binom{n-1}{q-1}\Big\{&K^{(q)}A^{(n-q)} + K^{(q-1)}A^{(n-q+1)} \\ &+ [K^{(q)}A^{(n-q)} + K^{(q-1)}A^{(n-q+1)}]^T \\ &- \sum_{q=2}^{n}\binom{n-1}{q-1}(KS)^{q-1}K^{(n-q+1)} \\ &- \sum_{q=1}^{n}\binom{n-1}{q-1}(KS)^{(q)}K^{(n-q)} \\ &+ C^{T(q)}QC^{(n-q)} + C^{T(q-1)}QC^{(n-q+1)}\Big\}\Big|_{\alpha=\alpha_o} \end{aligned} \quad (18a)$$

For $i<n$,

$$(KS)^{(i)} = \sum_{s=1}^{i}\binom{s-1}{i-1}\Big\{K^{(s)}S^{i-s} + K^{(s-1)}S^{(i-s+1)}\Big\} \quad (18b)$$

and for $i=n$,

$$(KS)^{(i)} = \sum_{s=1}^{i-1}\binom{s-1}{i-1}\Big\{K^{(s)}S^{i-s}\Big\} + \sum_{s=1}^{i}\binom{s-1}{i-1}\Big\{K^{(s-1)}S^{(i-s+1)}\Big\} \quad (18c)$$

with

$$S^{(j)} = \sum_{r=1}^{j}\binom{r-1}{j-1}\Big\{B^{(r)}R^{-1}B^{T(j-r)} + B^{(r-1)}R^{-1}B^{T(j-r+1)}\Big\} \quad (18d)$$

and

$$S \triangleq BR^{-1}B^T \quad (18e)$$

Solution of the recursive sensitivity equation (18a) is guaranteed by the following theorem.

Theorem 2: The Riccati derivative matrices will exist and be unique for all n in Eq. (18a), provided the following:

1) The plant is stabilizable and detectable for all admissible parameter perturbations.

2) The plant matrices are continuous and infinitely differentiable with respect to the parameters.

Proof of Theorem 2: Equation (18) is in the form

$$[K^{(n)}(\alpha)L(\alpha) + L^T(\alpha)K^{(n)}(\alpha)]_{\alpha=\alpha_o} = N(\alpha)|_{\alpha=\alpha_o} \tag{19}$$

Because the system is assumed stabilizable and detectable for all admissible parameter perturbations, the nominal closed loop is assured to be asymptotically stable, i.e., $\lambda_i[L(\alpha_o)]<0$ for $i = 1, 2,\dots,n_x$. The Lyapunov equation (19) is thus assured a unique solution for $K(\alpha)$ since[11]

$$\lambda_i[L(\alpha_o)] + \lambda_j[L(\alpha_o)] \neq 0 \qquad i,j = 1,2,\dots,n_x \tag{20}$$

Equation (18) is attractive for computer implementation because of its algorithmic form. Standard Lyapunov equation solution algorithms can be used to solve for each new sensitivity matrix. Each new Riccati sensitivity matrix is generated recursively from previous sensitivity matrices.

Vector Case

An algorithm incorporating recursive rule 1 is developed for repeated vector differentiation of a product of matrices. The algorithm requires the formation of a derivative operator matrix. Each row of the operator matrix is a product of scalar derivative operators defined by rule 1.

Recursive Product Rule 2 (Vector Differentiation of a Product of Matrices)

Consider the vector derivative of the matrix product

$$\frac{d^{(k)}}{d\alpha^{(k)}}\left[A(\alpha)B(\alpha)\right] \triangleq D_{\alpha^{(k)}}\left[A(\alpha)B(\alpha)\right] \tag{21}$$

where A(α) and B(α) are conformable for multiplication, $\alpha = [\alpha_1, \alpha_2,\dots \alpha_j,\dots \alpha_p]^T \in \mathbf{R}^{p\times 1}$, and A($\alpha$), B($\alpha$) are continuous and infinitely differentiable functions of α. We may write a recursive rule for repeated differentiation of the matrix product as

$$D_{\alpha^{(k)}}[A(\alpha)B(\alpha)] = \begin{matrix} \\ i\text{th row}\rightarrow \\ p^k\text{th row}\rightarrow \end{matrix}\begin{bmatrix} D_{\alpha_1}^{(\gamma_{1,1})}\cdots D_{\alpha_j}^{(\gamma_{1,j})}\cdots D_{\alpha_p}^{(\gamma_{1,p})} \\ D_{\alpha_1}^{(\gamma_{i,1})}\cdots D_{\alpha_j}^{(\gamma_{i,j})}\cdots D_{\alpha_p}^{(\gamma_{i,p})} \\ D_{\alpha_1}^{(\gamma_{p^k,1})}\cdots D_{\alpha_j}^{(\gamma_{p^k,j})}\cdots D_{\alpha_p}^{(\gamma_{p^k,p})} \end{bmatrix}[A(\alpha)B(\alpha)] \tag{22a}$$

The derivative-order indices $\gamma_{i,j}$ are defined as

$$\gamma_{i,j} = \sum_{q=1}^{K}\left\{\left(\mathrm{Int}\left[\frac{(i-1)\bmod p^{k-q+1}}{p^{k-q}}\right]+1\right)\stackrel{?}{=}j\right\} \quad \begin{matrix} i = 1,\dots,p^k \\ j = 1,\dots,p \end{matrix} \tag{22b}$$

with

$$\mathrm{Int}\,[\,\cdot\,] \triangleq \text{truncation to integer}$$

and "mod" is the modulo operator.

Each row in the derivative operator matrix is a product of scalar derivative operators that are defined by the rule in Eq. (22a). The following example illustrates the implementation of this formula.

Example 1:
Given: $A(\alpha), B(\alpha) \in \mathbf{R}^{3\times 3}$, $\alpha \in \mathrm{R}^{2\times 1}$
Find: $D_{\alpha^{(2)}}[A(\alpha)B(\alpha)]$

Using Eq. (22b), the γ_{ij} are computed as

$$\begin{aligned}
\gamma_{1,1} &= \{1 \stackrel{?}{=} 1\} + \{1 \stackrel{?}{=} 1\} = 2 \\
\gamma_{1,2} &= \{1 \stackrel{?}{=} 2\} + \{1 \stackrel{?}{=} 2\} = 0 \\
\gamma_{2,1} &= \{1 \stackrel{?}{=} 1\} + \{2 \stackrel{?}{=} 1\} = 1 \\
\gamma_{2,2} &= \{1 \stackrel{?}{=} 2\} + \{2 \stackrel{?}{=} 2\} = 1 \\
\gamma_{3,1} &= \{2 \stackrel{?}{=} 1\} + \{1 \stackrel{?}{=} 1\} = 1 \\
\gamma_{3,2} &= \{2 \stackrel{?}{=} 2\} + \{1 \stackrel{?}{=} 2\} = 1 \\
\gamma_{4,1} &= \{2 \stackrel{?}{=} 1\} + \{2 \stackrel{?}{=} 1\} = 0 \\
\gamma_{4,2} &= \{2 \stackrel{?}{=} 2\} + \{2 \stackrel{?}{=} 2\} = 2
\end{aligned}$$

The derivative operator matrix is developed via Eq. (22a) as

$$D_{\alpha^{(2)}}[A(\alpha)B(\alpha)] = \begin{bmatrix} D_{\alpha_1}^{(2)} \\ D_{\alpha_1}^{(1)}D_{\alpha_2}^{(1)} \\ D_{\alpha_1}^{(1)}D_{\alpha_2}^{(1)} \\ D_{\alpha_2}^{(2)} \end{bmatrix}[A(\alpha)B(\alpha)] \tag{23}$$

Completing the differentiation in Eq. (23) yields

$$D_{\alpha^{(2)}}[A(\alpha)B(\alpha)] = \begin{bmatrix} A_{\alpha_1}^{(2)}B + 2A_{\alpha_1}^{(1)}B_{\alpha_1}^{(1)} + AB_{\alpha_1}^{(2)} \\ A_{\alpha_1\alpha_2}^{(1)(1)}B + A_{\alpha_2}^{(1)}B_{\alpha_1}^{(1)} + A_{\alpha_1}^{(1)}B_{\alpha_2}^{(1)} + AB_{\alpha_1\alpha_2}^{(1)(1)} \\ A_{\alpha_1\alpha_2}^{(1)(1)}B + A_{\alpha_1}^{(1)}B_{\alpha_2}^{(1)} + A_{\alpha_2}^{(1)}B_{\alpha_1}^{(1)} + AB_{\alpha_1\alpha_2}^{(1)(1)} \\ A_{\alpha_2}^{(2)}B + 2A_{\alpha_2}^{(1)}B_{\alpha_1}^{(1)} + AB_{\alpha_2}^{(2)} \end{bmatrix} \tag{24}$$

This derivative operator notation developed in Eq. (24) applied to Eq. (8e) yields

$$0 = \left(D_\alpha^{(n)}\left[K(\alpha)A(\alpha)\right]\right) + \left(D_\alpha^{(n)}\left[K(\alpha)A(\alpha)\right]^{\mathrm{T}}\right) - D_\alpha^{(n)}\left[\left(K(\alpha)S(\alpha)\right)K(\alpha)\right] + D_\alpha^{(n)}\left[C^T(\alpha)QC(\alpha)\right] \tag{25}$$

Equation (25) may be expanded using the recursive product rules 1 and 2, and reorganized to yield a matrix equation of the following form:

$$[K']L + (L^T \times I_{p^n})[K'] = [N(\alpha)] \tag{26}$$

The ith $n\times n$ partition of $[K']$ is defined as

$$[K']_i \triangleq (D_{\alpha_1}^{(\gamma_{i,1})}\cdot D_{\alpha_2}^{(\gamma_{i,2})}\cdots D_{\alpha_p}^{(\gamma_{i,p})})K \tag{27}$$

$[N(\alpha)]$ represents all remaining terms resulting from the expansion of Eq. (26) not multiplying $L(\alpha_o)$ or $L^T(\alpha_o)$.

Each term in vector equation (26) reduces to an $n_x p^n \times n_x$ matrix. The equations may be partitioned by rows into $p^n n_x \times n_x$ subequations. Each of the subequations admit a Lyapunov form similar to Eq. (19). The solution procedure for the vector equation proceeds analogously to the scalar case. At each step of the recursive differentiation, a total of $\binom{p^n}{n}$ Lyapunov equations of order n_x must be computed. The existence and uniqueness properties of Theorem 2 may be extended to each subequation, thereby establishing the existence and uniqueness for the vector case.

The general nth order Taylor series expansion of the regulator gain matrix dependent on the parameter vector α can be expressed using a vector form of the Taylor series given by Vetter.[12]

$$^{n}G(\alpha) = G(\alpha_o) + R^{-1}\sum_{k=1}^{n}\frac{1}{k!} \times \left(\overset{\otimes k}{(\alpha-\alpha_o)^T} \otimes \underset{p}{I}\right) D_\alpha^{(k)}\left(B^T(\alpha)K(\alpha)\right) \tag{28}$$

where

$$^{\otimes}\xi^k \triangleq \underbrace{\otimes\ \xi \cdots \otimes\ \xi}_{k \text{ times}}$$

Using the recursive differentiation algorithms developed previously and exploiting the structure of the Riccati sensitivity equations, it is possible to generate general-purpose software that will compute the gain-sensitivity matrices to an arbitrary order. The duality principle[11] permits the same computer algorithms developed for the regulator design to be used for the filter design. The major computational task is associated with the tedious bookkeeping required to track intermediate results. For large systems with several parameter dependencies, the computational burden may become large, however, with efficient programming, only moderate memory is required.

Convergence Properties and Performance Bounds for GSA-LQG Control Systems

The convergence of Taylor series expansions for LQG gain matrices is not guaranteed. It was shown by Theorem 2 that Riccati derivatives matrices of all orders exist for the stated assumptions; however, infinite differentiability of a real continuous function does not insure existence of a convergent Taylor series (Ref. 13, p. 691). Indeed, even if a convergent Taylor series exists for a real function about some point, it may have finite radius of convergence. The difficulty of a limited radius of convergence can be circumvented by construction of a schedule of gain-sensitivity augmented feedback laws. The authors' experiences indicate that, for well-posed problems, convergent Riccati matrix expansions that span the set of admissible parameter perturbations can always be constructed.

Performance properties of the augmented control system are examined next. A primary objective of the GSA control design is to maintain near-optimal performance with respect to the cost function. Of interest is the performance degradation induced by the suboptimal controller. Calculation of the bound is complicated by the parameter dependence of the system matrices. First, the following fact is noted.

Lemma 1[14]:
For $A \geq 0, \quad A = A^T, \quad B$ real

$$\operatorname{tr}(AB) \leq \left|\,|B|\,\right|_s \operatorname{tr}(A) \tag{29}$$

Let n denote the order of gain-sensitivity augmentation. From the assumptions in the problem formulation, we can state the following theorem.

Theorem 3: If $F(\alpha)$ and $G(\alpha)$ admit convergent Taylor series expansions for $\alpha \in \Omega$, then $\hat{A}$ yields a stability matrix for some n, where $0 < n < \infty$,

$$\hat{A} \triangleq \begin{bmatrix} A(\alpha) + B(\alpha)\hat{G}(\alpha) & -B(\alpha)\hat{G}(\alpha) \\ 0 & A(\alpha) - \hat{F}(\alpha)M(\alpha) \end{bmatrix} \tag{30a}$$

$$\hat{G} \triangleq G(\alpha) + \Delta G(\alpha) \tag{30b}$$

$$\hat{F} \triangleq F(\alpha) + \Delta F(\alpha) \tag{30c}$$

Proof of Theorem 3: To prove this thorem, it is sufficient to demonstrate the system $\dot{x} = \hat{A}x$ is asymptotically stable.

Define:

$$\hat{A} \triangleq A + \Delta A \tag{31}$$

where

$$\Delta A \triangleq \begin{bmatrix} B\Delta G & -B\Delta G \\ 0 & -\Delta FM \end{bmatrix} \tag{32}$$

Given: $P > 0$, $P = P^T$, and Z, which solves the Lyapunov equation

$$\hat{A}^T Z + Z\hat{A} + P = 0 \tag{33}$$

then, $\hat{A}$ is a stability matrix if (Ref. 9, p. 513)

$$P - \Delta A^T Z - Z\Delta A > 0 \tag{34a}$$

$$\rightarrow P > \Delta A^T Z + Z\Delta A \tag{34b}$$

We may write

$$2\left|\,|\Delta A|\,\right| \left|\,|Z|\,\right| I_n \geq \Delta A^T Z + Z\Delta A \tag{35}$$

A more conservative condition than Eq. (34b) is

$$P \geq 2\left|\,|\Delta A|\,\right| \left|\,|Z|\,\right| I_{2n_x} \tag{36}$$

By assumption,

$$\lim_{n\to\infty} \left|\,|\Delta F|\,\right| = 0 \tag{37a}$$

$$\lim_{n\to\infty} \left|\,|\Delta G|\,\right| = 0 \tag{37b}$$

$$=> \lim_{n\to\infty} \left|\,|\Delta A|\,\right| = 0 \tag{37c}$$

thus

$$P > \lim_{n\to\infty}\left\{2\left|\,|\Delta A|\,\right| \left|\,|Z|\,\right| I_{n_x}\right\} \tag{38}$$

Theorem 3 insures that a stabilizing controller for some finite augmentation order exists for the prescribed conditions. Theorem 4, given next, quantifies the performance degradation associated with a finite augmentation order.

Theorem 4: The steady-state cost degradation for the stable suboptimal GSA-LQG system, given the plant in Eq. (6a)-(6d), and the controller in Eqs. (12a) and (12b) is bounded by

$$\left|\,|\Delta \mathbf{V}|\,\right|_s \leq \left(2\left|\,|\tilde{K}|\,\right|_s \left|\,|\Delta A|\,\right|_s + \left|\,|\Delta\tilde{D}|\,\right|_s\right) \left|\,|K_I|\,\right|_s \left|\,|(DWD^T)^{1/2}|\,\right|_F + \left|\,|(\tilde{K})^{1/2}|\,\right|_F \left|\,|\Delta\tilde{D}|\,\right|_s \tag{39}$$

where

$$\Delta\tilde{D} \triangleq \tilde{D}W\tilde{D}^T - DWD^T \tag{40a}$$

$$\tilde{D} \triangleq \begin{bmatrix} D & 0 \\ D & -\hat{F}(\alpha) \end{bmatrix} \tag{40b}$$

and

$$0 = \hat{K}\hat{A} + \hat{A}^T\hat{K} + \hat{D}W\hat{D}^T \quad (41a)$$

$$0 = KA + A^TK + DWD^T \quad (41b)$$

$$0 = K_IA + AK_I + I_{2n_x} \quad (41c)$$

Proof of Theorem 4:
Define

$$\Delta\mathcal{V} = \hat{\mathcal{V}} - \mathcal{V} \geq 0 \quad (42)$$

where

$$\hat{\mathcal{V}} = \text{tr}\ \hat{K}\hat{D}W\hat{D}^T \quad (43)$$

$$\mathcal{V} = \text{tr}\ KDWD^T \quad (44)$$

Then,

$$\Delta\mathcal{V} = \text{tr}\Delta KDWD^T + \text{tr}\hat{K}\ \Delta\tilde{D} \quad (45)$$

where $\Delta K \triangleq \hat{K} - K$. By Lemma 1, we have

$$\Delta\mathcal{V} \leq \left|\left|\Delta K\right|\right|_s \text{tr}(DWD^T) + \text{tr}\hat{K}\left|\left|\Delta\tilde{D}\right|\right|_s \quad (46)$$

Because $DWD^T \geq 0$, $\hat{K} \geq 0$,

$$\text{tr}\ DWD^T = \left|\left|(DWD^T)^{1/2}\right|\right|_F \quad (47)$$

$$\text{tr}\hat{K} = \left|\left|(\hat{K})^{1/2}\right|\right|_F \quad (48)$$

Thus,

$$\Delta\mathcal{V} \leq \left|\left|\Delta K\right|\right|_s \left|\left|(DWD^T)^{1/2}\right|\right|_F + \left|\left|\hat{K}^{1/2}\right|\right|_F \left|\left|\Delta\tilde{D}\right|\right|_s \quad (49)$$

We can show

$$\left|\left|\Delta K\right|\right|_s \leq \left(2\left|\left|\hat{K}\right|\right|_s \left|\left|\Delta A\right|\right|_s + \left|\left|\Delta\tilde{D}\right|\right|_s\right)\left|\left|K_I\right|\right|_s \quad (50)$$

From Eq. (41a), we can write

$$\hat{K}A + A^T\hat{K} = -\hat{D}W\hat{D}^T - \hat{K}\Delta A - \Delta A^T\hat{K} \quad (51a)$$

For a stable A, we can then write

$$\hat{K} = \int_0^\infty \exp(At)[-\hat{K}\Delta A - \Delta A^T\hat{K}\hat{D}W\hat{D}]\exp(A^Tt)\ dt \quad (51b)$$

From Eq. (41b), we can write

$$K = \int_0^\infty \exp(At)DWD^T\exp(A^Tt)\ dt \quad (52)$$

Subtracting Eq. (52) from Eq. (51b) yields

$$\Delta K = -\int_0^\infty \exp(At)[\Delta\tilde{D} + \hat{K}\Delta A + A^T\hat{K}]\exp(A^Tt)\ dt \quad (53)$$

To simplify the result in Eq. (53), we examine the integrand. For any symmetric $s \times s$ matrix, we have

$$N \leq \left|\left|N\right|\right| I_s \quad (54)$$

Thus, we may rewrite the integrand in Eq. (53) as

$$\Delta\tilde{D} + \hat{K}\Delta A + \Delta A^T\hat{K} \leq \left(2\left|\left|\hat{K}\right|\right|_s \left|\left|\Delta A\right|\right|_s + \left|\left|\Delta\tilde{D}\right|\right|_s\right) I_{2n_x} \quad (55)$$

So Eq. (53) is rewritten as

$$\left|\left|\Delta K\right|\right|_s \leq \left(2\left|\left|\hat{K}\right|\right|_s \left|\left|\Delta A\right|\right|_s + \left|\left|\Delta\tilde{D}\right|\right|_s\right)\int_0^\infty \exp(At)\exp(A^Tt)\ dt \quad (56)$$

But

$$K_I \triangleq \int_0^\infty \exp(At)\exp(A^Tt)\ dt \quad (57)$$

where K_I is defined in Eq. (41c).

Substituting Eq. (57) in Eq. (56) yields Eq. (49), and then substituting Eq. (50) in Eq. (49) results in Eq. (39).

From the cost degradation bound given by Theorem 4, the convergence of the GSA-LQG controller to the optimal parameter-dependent LQG controller can be proved.

Theorem 5: If F (α) and G (α) admit convergent Taylor series expansion for $\alpha \in \Omega$, then,

$$\lim_{n\to\infty}\left|\left|\Delta\mathcal{V}\right|\right|_s = 0 \quad (58)$$

Proof of Theorem 5: From Theorem 3, we know that as $n \to \infty$, where n is the augmentation order, $\hat{A}$ becomes a stability matrix. Thus, the bound given in Eq. (39) exists and is finite. We simply note that, by virtue of Eq. (37a) and (37c),

$$\lim_{n\to\infty}\left|\left|\Delta A\right|\right|_s = 0 \quad (59)$$

$$\lim_{n\to\infty}\left|\left|\Delta\tilde{D}\right|\right|_s = 0 \quad (60)$$

Thus, Eq. (60) is implied from Eq. (38).

Controller Realization

Figure 1 shows a block diagram of the closed-loop control system using gain-sensitivity augmented feedback. The param-

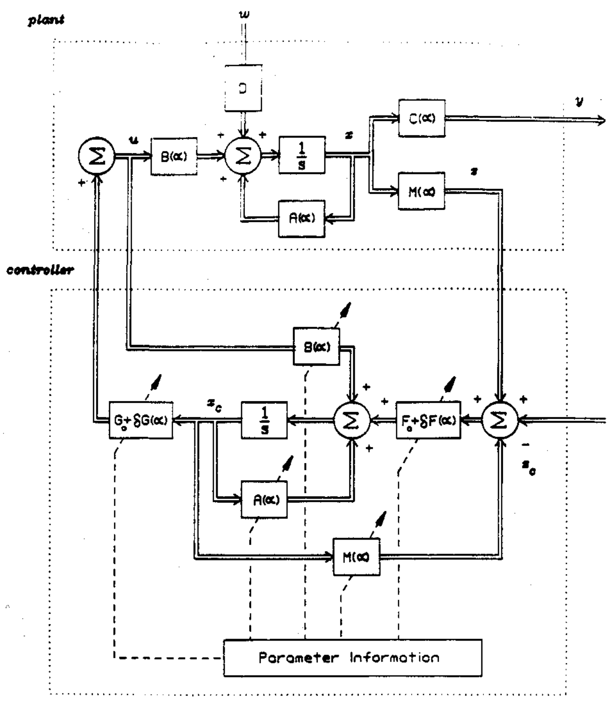

Fig. 1 The GSA-LQG controller.

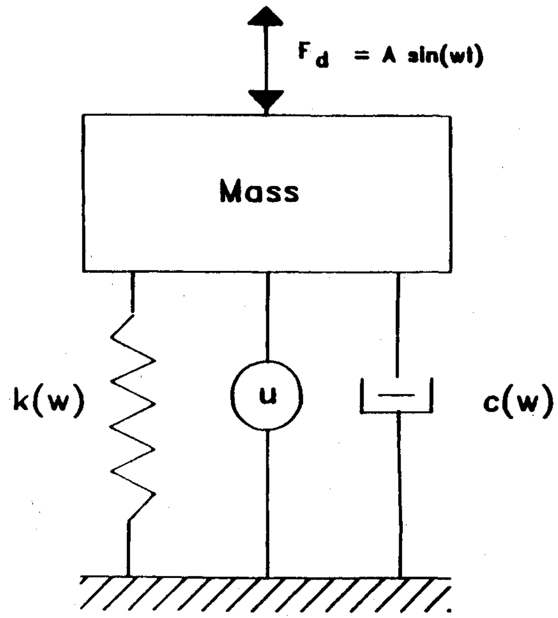

Fig. 2 Vibration-control example.

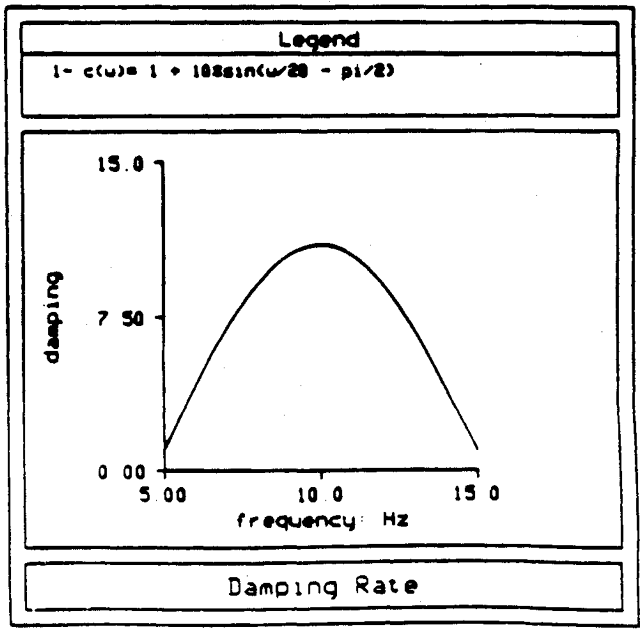

Fig. 4 Damping characteristic of machine mounting.

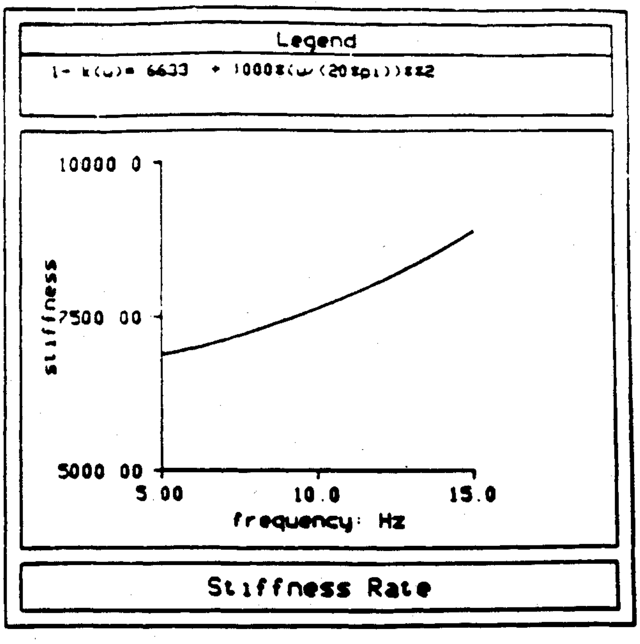

Fig. 3 Stiffness characteristic of machine mounting.

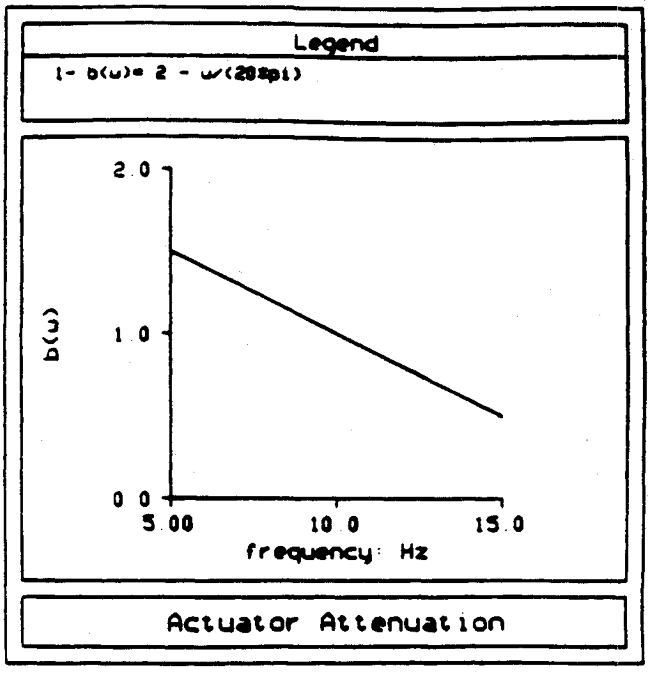

Fig. 5 Actuator attenuation characteristic.

eter feedback is used to adjust the nominal regulator and filter feedback gain matrices to near-optimal values. The parameters are also used for on-line adjustment of the Kalman filter equations to improve estimation of plant states.

A key remaining design task is determining the truncation order of the gain-sensitivity augmentation. Considerations such as stability, performance degradation, and complexity influence the decision process. The open literature suggests various stability tests for parameter-dependent systems; however, these tend to be conservative. Theorem 4 provides a measure for performance degradation, but this metric is not computationally practical. The authors suggest design by simulation to determine adequate gain-sensitivity augmentation.

An example is given next illustrating the gain-sensitivity augmentation method applied to a plant with a complicated scalar parameter dependency.

Example: Consider a reciprocating machine mounted on a passive isolation system. A force-generating actuator is attached between the machine and ground. The passive isolation pad has frequency-dependent stiffness and damping properties. The actuator is modeled with frequency-dependent attenuation. The dominant vibration resulting from the reciprocating components in the machine occurs at its operating frequency and is harmonic. The machine is assumed to operate at speeds of 5–15 Hz. The nominal operating speed is taken to be 10 Hz. It is desired to design an active vibration control system over the working speed range of the engine. We will examine controller performance when perturbing the engine speed from 10 to 15 Hz.

Figure 2 shows a 1-DOF model for the vibration system. The system is described by a fourth-order model. The model contains a second-order disturbance model that is parameter-

Table 1 Augmented regulator gain matrices

Augmentation order	Gain matrices			
Nominal	$-6.551e-02$	$-2.248e+01$	$-1.659e-01$	$4.539e-03$
1	$-2.417e-02$	$-1.879e+01$	$-4.250e-01$	$-5.447e-03$
2	$-2.744e-02$	$-1.879e+01$	$-4.250e-01$	$5.447e-03$
3	$-2.836e-02$	$-2.742e+01$	$-1.015e+00$	$8.484e-03$
4	$-2.814e-02$	$-2.796e+01$	$-1.309e+00$	$8.641e-03$
5	$-2.814e-02$	$-2.902e+01$	$-1.575e+00$	$8.224e-03$
6	$-2.814e-03$	$-2.931e+01$	$-1.797e+00$	$7.248e-03$
7	$-2.814e-02$	$-2.945e+01$	$-1.967e+00$	$5.931e-03$
8	$-2.814e-02$	$-2.960e+01$	$-2.086e+00$	$4.451e-03$
9	$-2.814e-02$	$-2.964e+01$	$-2.154e+00$	$2.946e-03$
10	$-2.814e-02$	$-2.966e+01$	$-2.178e+00$	$1.546e-03$
Target	$-2.814e-02$	$-2.969e+01$	$-1.802e+00$	$3.874e-06$

Table 2 Augmented filter gain matrices

Augmentation order	Gain matrices			
Nominal	$0.000e+00$	$3.979e+01$	$2.294e+02$	$1.769e+04$
1	$0.000e+00$	$3.979e+01$	$2.295e+02$	$3.733e+02$
2	$0.000e+00$	$4.919e+01$	$2.157e+02$	$2.379e+02$
3	$0.000e+00$	$4.919e+01$	$2.158e+02$	$2.377e+02$
4	$0.000e+00$	$4.881e+01$	$2.261e+02$	$2.401e+02$
5	$0.000e+00$	$4.881e+01$	$2.261e+02$	$2.334e+02$
6	$0.000e+00$	$4.843e+01$	$2.232e+02$	$2.269e+02$
7	$0.000e+00$	$4.843e+01$	$2.232e+02$	$2.264e+02$
8	$0.000e+00$	$4.846e+01$	$2.233e+02$	$2.286e+02$
9	$0.000e+00$	$4.846e+01$	$2.233e+02$	$2.312e+02$
10	$0.000e+00$	$4.848e+01$	$2.234e+02$	$2.327e+02$
Target	$0.000e+00$	$4.848e+01$	$2.234e+02$	$2.312e+02$

ized by the machine speed (ω). The disturbance model accounts for the harmonic excitation force F_d. It is assumed that the vertical velocity of the machine and the angular velocity (ω) of the reciprocating components are measurable.

The model of the system is described as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{-k(\omega)}{m} & \dfrac{-c(\omega)}{m} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & -\epsilon \end{bmatrix} x + \begin{bmatrix} 0 \\ b(\omega) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u + w \end{bmatrix} \tag{61a}$$

$$y = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{bmatrix} x \tag{61b}$$

$$z = [0\ 1\ 0\ 0]\ x + v \tag{61c}$$

$$u = {}^nG(\omega)x_c \tag{61d}$$

$$F_d = A \sin\omega t \tag{61e}$$

$$w \sim N[0,1] \tag{61f}$$

$$v \sim N[0,0.005] \tag{61g}$$

$$\epsilon = 0.62 \tag{61h}$$

The performance criterion is given as

$$\mathcal{V} = \lim_{n\to\infty} E\left[\frac{1}{t}\int_0^\infty x_1^2 + x_2^2 + \frac{u^2}{1000}\,\mathrm{d}t\right] \tag{62}$$

The parameter ω is constrained to the following set:

$$\omega \in [10\pi,\ 30\pi], \qquad \omega_o = 20\pi \text{ rad/s} \tag{63}$$

The equations describing the frequency-dependent characteristic of the stiffness, damping, and actuator elements follow. Figures 3–5 show these characteristics. The mass is taken to be unity.

$$k(\omega) = 6633 + 1000\left[\frac{\omega}{\omega_o}\right]^2 \tag{64}$$

$$c(\omega) = 1 + 10\sin\left[\frac{\omega}{\omega_o} - \frac{1}{2}\right]\pi \tag{65}$$

$$b(\omega) = 2 - \frac{\omega}{\omega_o} \tag{66}$$

The disturbance dynamics are incorporated into the plant model via the lower 2×2 partition of the A matrix, permitting control design with optimal disturbance utilization. The optimal control design incorporates a Kalman filter, which estimates the plant states and the disturbance states. The feedback of the disturbance state estimates is equivalent to feedforward compensation.

Table 1 shows a comparison of the regulator gain matrices for the ω perturbed to 15 Hz. The augmentation order ranges from 1 to 10. Table 2 contains the Kalman filter gain matrices. A right shift was imposed on the A matrix in the filter design to improve the disturbance estimation. The right shift used was

$$A_{\text{shift}} = [A] + \begin{bmatrix} 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 \\ 0\,0 & 1.5 & 0 \\ 0\,0 & 0 & 1.5 \end{bmatrix} \tag{67}$$

Figure 6 contrasts the nominal closed-loop performance with the open-loop performance. Figure 7 compares the regulation performance of the nominal controller, first- and fifth-order augmented controllers, and the target controller. The target controller is defined as the optimal controller for the perturbed system. Figure 8 compares the performance of the disturbance estimation for these same controllers. Note the poor estimation by the nominal controller. Figure 9 shows the actuator control force for the different controllers.

Figure 10 compares the performance cost for regulator designs with augmentation order from 1 to 10. Augmentation order 0 corresponds to the nominal controller, whereas order 11 corresponds to the target controller. The performance costs were computed by simulation to steady state of the forced system. A performance improvement of nearly 100% was achieved with the first-order augmentation. An additional improvement of 20% is achieved with augmentation to the fifth order. This example demonstrates that, for large parameter perturbations, GSA controller design may yield significant improvement in closed-loop performance.

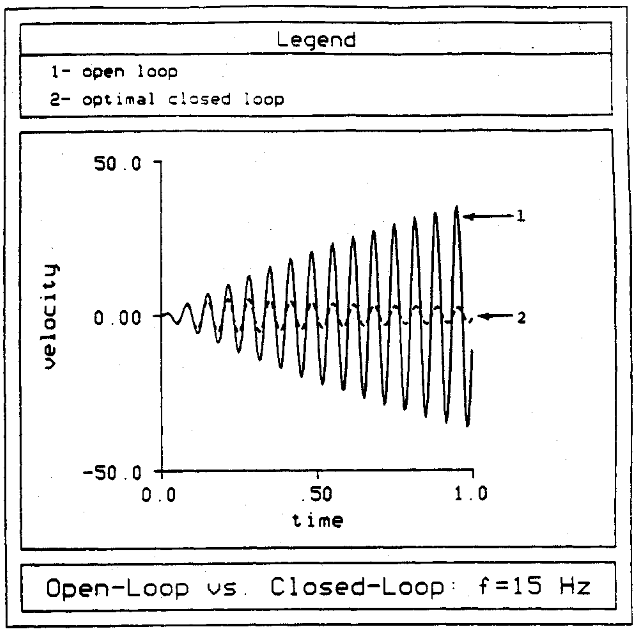

Fig. 6 Open- vs closed-loop response.

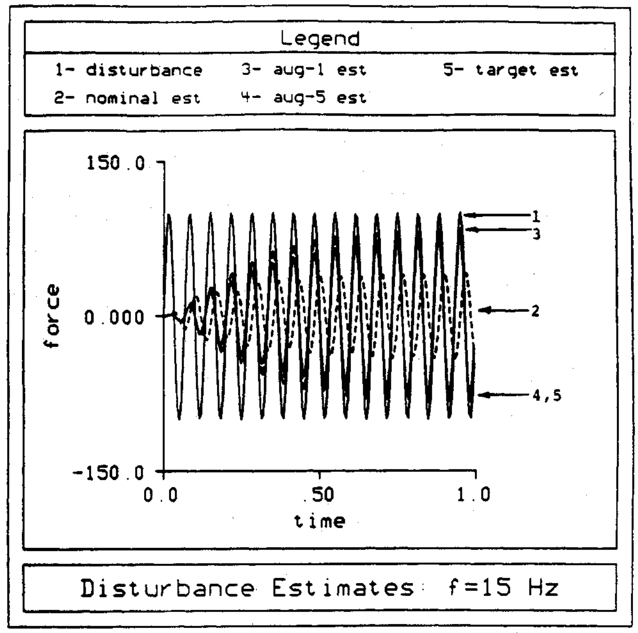

Fig. 8 Comparison of disturbance estimation.

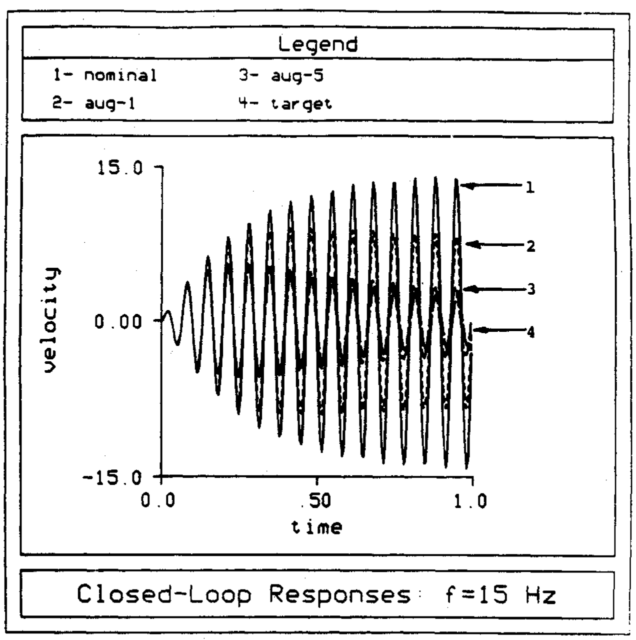

Fig. 7 Comparison of controller regulation.

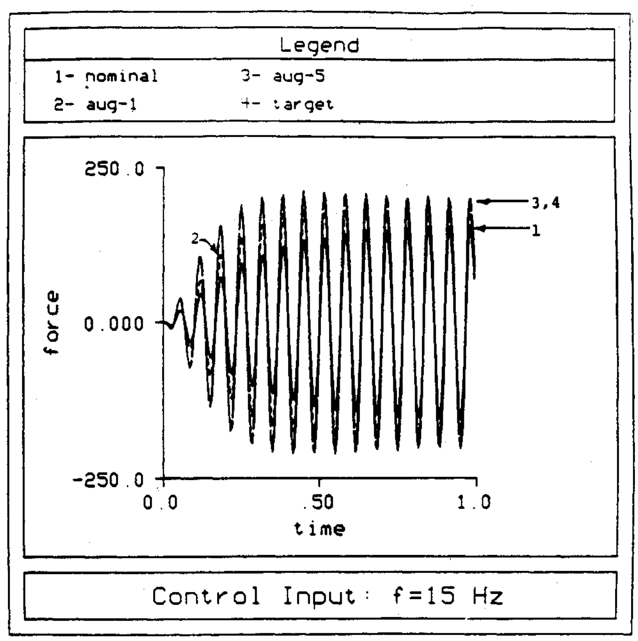

Fig. 9 Control input.

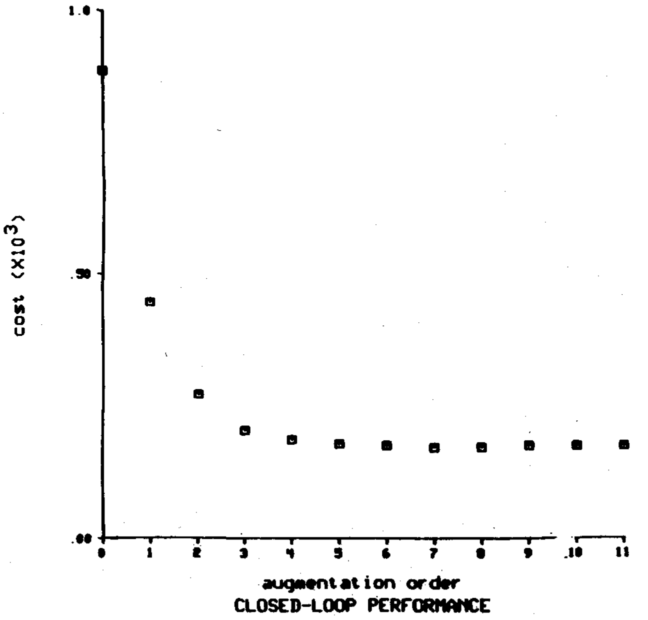

Fig. 10 Cost comparison of regulator performance.

Conclusions

Many physical systems can be represented by linear parameter-dependent plants, whereas parameter variations are significantly large. This paper presented a technique for closed-loop feedback control of such systems. The results indicated that the proposed feedback augmentation technique develops a near-optimal solution to the parameter-dependent systems. It is concluded that, with a few orders of augmentation, performance of the closed-loop parameter-varying system matches very closely the optimal target performance of a parameter-invariant system. Furthermore, with the addition of this augmentation technique to the observer feedbacks, accurate disturbance estimation can be achieved. Currently, this technique is being studied for application in semiactive/active control of flexible structures.

Appendix

Proof of Theorem 1: The closed-loop composed of the plant in Eq. (6a–6c) and the controller structure of Eq. (8a–8f) may be combined as follows:

$$\dot{\boldsymbol{x}}(t,\alpha) = \boldsymbol{A}\boldsymbol{x}(t\alpha) + \boldsymbol{D}\boldsymbol{w}(t) \tag{A1a}$$

$$\boldsymbol{y}(t,\alpha) = \boldsymbol{C}(t,\alpha)\boldsymbol{x}(t,\alpha) \tag{A1b}$$

where

$$\boldsymbol{x}(t,\alpha) \triangleq \begin{bmatrix} x(t,\alpha) \\ x(t,\alpha) - x_c(t,\alpha) \end{bmatrix};$$

$$\boldsymbol{y}(t\alpha) \triangleq \begin{bmatrix} y(t,\alpha) \\ u(t,\alpha) \end{bmatrix}, \qquad \boldsymbol{w}(t) \triangleq \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \tag{A1c}$$

$$\boldsymbol{A} = \begin{bmatrix} A(\alpha) + B(\alpha)G(\alpha) & -B(\alpha)G(\alpha) \\ 0 & A(\alpha) - F(\alpha)M(\alpha) \end{bmatrix};$$

$$\boldsymbol{D} = \begin{bmatrix} D & 0 \\ D & -F(\alpha) \end{bmatrix} \tag{A1d}$$

$$\boldsymbol{C} = \begin{bmatrix} C(\alpha) & 0 \\ G(\alpha) & -G(\alpha) \end{bmatrix} \tag{A1e}$$

The steady-state cost for the closed-loop system may be written as

$$\mathcal{V} = \text{tr}\boldsymbol{X}\boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C} \tag{A2}$$

with

$$0 = \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{X} + \boldsymbol{D}\boldsymbol{W}\boldsymbol{D}^T \tag{A3}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}, \qquad \boldsymbol{W} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \tag{A4}$$

The optimal control problem will now be formulated by appealing to Hamilton's principle. Introducing Lagrange multipliers in the form of the symmetric matrix Λ, the Hamiltonian may be written as

$$H = \text{tr}\left(\boldsymbol{X}\boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C} + \Lambda(\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A}^T + \boldsymbol{D}\boldsymbol{W}\boldsymbol{D}^T)\right) \tag{A5}$$

The minimization is performed relative to gain matrices G and F, the state covariance X, and the Lagrange multiplier matrix Λ. The parameters α are considered specified and therefore do not contribute additional constraints to the objective function. The optimization problem may be written as

$$\min_{G,F,X,\Lambda} H \tag{A6a}$$

To derive the first-order necessary conditions, the following trace differentiation identities are required:

$$\text{tr}(AB) = \text{tr}(BA) = \text{tr}(A^TB^T) \tag{A6b}$$

$$\frac{\text{d}}{\text{d}A}\,\text{tr}(AB) = B^T \tag{A6c}$$

$$\frac{\text{d}}{\text{d}B^T}(B^TA^T) = A \tag{A6d}$$

Define

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}; \qquad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} \tag{A7}$$

The first-order necessary conditions are

$$\frac{\partial H}{\partial X} = 0 = \Lambda\boldsymbol{A} + \boldsymbol{A}^T\Lambda + \boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C} \tag{A8}$$

$$\frac{\partial H}{\partial \Lambda} = 0 = \boldsymbol{X}\boldsymbol{A}^T + \boldsymbol{A}\boldsymbol{X} + \boldsymbol{D}\boldsymbol{W}\boldsymbol{D}^T \tag{A9}$$

$$\frac{\partial H}{\partial G} = 0 = 2G(X_{11} + X_{22} - X_{12} - X_{12}^T) + 2B^T\left(\Lambda_{11}(X_{11} - X_{12}) + \Lambda_{12}(X_{12}^T - X_{22})\right) \tag{A10}$$

$$\frac{\partial H}{\partial F} = 0 = 2\Lambda_{22}FV - [2\Lambda_{12}^TX_{12} + 2\Lambda_{22}X_{22}]M^T \tag{A11}$$

It can be shown by direct substitution and by exploiting the existence and uniqueness properties of Lyapunov equations that the definitions for X and Λ given in Eqs. (12a) and (12b) solve Eqs. (A8–A11) uniquely subject to the assumptions of the problem formulation. Equations (8a–8f) may be recovered by suitable reorganization of Eqs. (A8–A11). Thus, the necessary conditions for optimal control are satisfied. Sufficiency is satisfied by R, $V > 0$.

$$X_{12} = X_{12}^T = X_{22} = P \tag{A12a}$$

$$\Lambda_{12} = \Lambda_{12}^T = 0 \tag{A12b}$$

The optimal cost that minimizes Eq. (7) may be expressed as

$$\mathcal{V} = \mathrm{tr} XC^TQC = \mathrm{tr}\,[X_{11}C^TQC + (X_{11} - P)G^TRG] \tag{A13}$$

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